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THESIS

TIME-DOMAIN DESIGN OF OBSERVERS

by

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Time-Domain Design of Observers

by

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ABSTRACT

A new technique for observer design which requires the solution of a minimax problem is presented. An algorithm for solving the minimax problem is proposed and applied to the design of full-order and reduced-order observers.

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I. INTRODUCTION

In feedback control systems, the input vector to a plant is often a function of the current plant state vector. This means that one requires knowledge of the plant state vector at the current time to accomplish system control. In most systems, however, the entire state vector is not available for direct measurement. Hence, it is desirable to reconstruct, or estimate, the unknown plant state vector. The device which performs this reconstruction of the plant state vector is called an "observer." In this thesis, after presenting necessary background material on the theory of observers, which is primarily the work of Luenberger [4], a new approach to observer design is systematically developed. This new method uses a quadratic performance measure which is to be minimized with respect to the observer gain matrix $\underline{G}$.

For simplicity, only linear time-invariant continuous-time systems are considered throughout this thesis.

The algorithm proposed is suitable for digital computation using appropriate subroutines and is applicable to higher-order systems. In order to illustrate the new technique, numerical examples are presented.

II. OBSERVER THEORY

A. THE NEED FOR OBSERVERS

To design a feedback control system it is often desired to know the entire plant state vector. But, in most cases, the entire state vector cannot be measured, hence a suitable estimate to the plant state vector must be determined. This creates the need for a system called an "observer" which produces an estimate of the plant state vector.

Assuming that it is possible to build a system which can estimate some constant linear transformation of the system state vector, then, provided that the transformation is invertible, it is possible to reconstruct the state vector itself. This is the fundamental idea upon which observer theory is based.

B. FULL-ORDER OBSERVERS

First consider the construction of a full-order observer for a free system described by

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) \qquad (1)$$

$$\underline{y}(t) = \underline{C}\ \underline{x}(t) \qquad (2)$$

where $\underline{x}(t)$ is the nxl state vector, $\underline{y}(t)$ is the mxl output vector, $\underline{A}$ is an nxn constant matrix and $\underline{C}$ is an mxn constant matrix, and $n > m$. Thus, an m-dimensional output vector $\underline{y}(t)$ is available from measurements, but we want to know the entire state vector $\underline{x}(t)$ of the plant. Since $\underline{x}(t)$ cannot be

measured directly, one must obtain an estimate $\hat{\underline{x}}(t)$, that is a good approximation to the original plant state $\underline{x}(t)$ for all time. In what follows it is assumed that the plant described by (1) and (2) is observable.

One way to estimate the state vector $\underline{x}(t)$ might be to construct a system of the same dynamic order and structure as the plant, given by

$$\dot{\underline{z}}(t) = \underline{A}\ \underline{z}(t) \tag{3}$$

and drive it in parallel with the plant. If this is done the system state, $\underline{z}(t)$, will be a linear transformation of the plant state $\underline{x}(t)$. If the initial states $\underline{z}(0)$ and $\underline{x}(0)$ are equal, then $\underline{z}(t)$ and $\underline{x}(t)$ remain equal for all $t \geq 0$. Usually, however, $\underline{z}(0)$ is not equal to $\underline{x}(0)$ and the difference between the two states $\underline{z}(t)$ and $\underline{x}(t)$, can be determined by forming

$$\dot{\underline{z}}(t) - \dot{\underline{x}}(t) = \underline{A}\ [\underline{z}(t) - \underline{x}(t)] \tag{4}$$

The solution of (4) is

$$\underline{z}(t) - \underline{x}(t) = \varepsilon^{\underline{A}t}\ [\underline{z}(0) - \underline{x}(0)] \tag{5}$$

Thus, the error $\underline{z}(t) - \underline{x}(t)$ depends on the matrix $\underline{A}$. If the matrix $\underline{A}$ is a stability matrix, i.e., all the eigenvalues of the matrix $\underline{A}$ have negative real parts, the error approaches zero as time goes to infinity.

A better alternative is to construct a linear system which contains a model of the plant state equation but is

also driven by an output error signal, $\underline{y}(t) - \underline{C}\ \underline{z}(t)$, that is,

$$\dot{\underline{z}}(t) = \underline{A}\ \underline{z}(t) + \underline{G}[\underline{y}(t) - \underline{C}\ \underline{z}(t)] \tag{6}$$

or

$$\dot{\underline{z}}(t) = (\underline{A} - \underline{G}\ \underline{C})\ \underline{z}(t) + \underline{G}\ \underline{y}(t) \tag{7}$$

This system has a very desirable feature: the observer gain matrix $\underline{G}$ can be selected to determine the rate at which the observer state $\underline{z}(t)$ approaches the plant state $\underline{x}(t)$. If the state $\underline{z}(t)$ becomes equal to the plant state $\underline{x}(t)$ at some time $t = t_1$, then the correction term, $\underline{y}(t) - \underline{C}\ \underline{z}(t)$, is zero for $t = t_1$ and the state $\underline{z}(t)$ will remain aligned with the plant state $\underline{x}(t)$ for $t \geq t_1$.

The system described by equation (6) or (7) is referred to as a "full-order observer" for the plant described by (1) and (2). The gain matrix $\underline{G}$ can be selected by the designer. How to select the gain matrix $\underline{G}$ is the problem of observer design.

The above development can be easily extended to forced systems by including the input $\underline{u}(t)$ in the observer equation (6) or (7). Consider the system described by

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{u}(t) \tag{8}$$

$$\underline{y}(t) = \underline{C}\ \underline{x}(t) \tag{9}$$

where $\underline{u}(t)$ is the rx1 input vector and $\underline{B}$ is an nxr constant distribution matrix. The observer for this plant is similar to that for the free system, but the input vector $\underline{u}(t)$ must be included. The appropriate full-order observer system is

$$\dot{\underline{z}}(t) = \underline{A}\ \underline{z}(t) + \underline{B}\ \underline{u}(t) + \underline{G}[\underline{y}(t) - \underline{C}\ \underline{z}(t)] \qquad (10)$$

or

$$\dot{\underline{z}}(t) = (\underline{A} - \underline{G}\ \underline{C})\ \underline{z}(t) + \underline{B}\ \underline{u}(t) + \underline{G}\ \underline{y}(t) \qquad (11)$$

The configuration of this full-order observer is shown in Fig. 1.

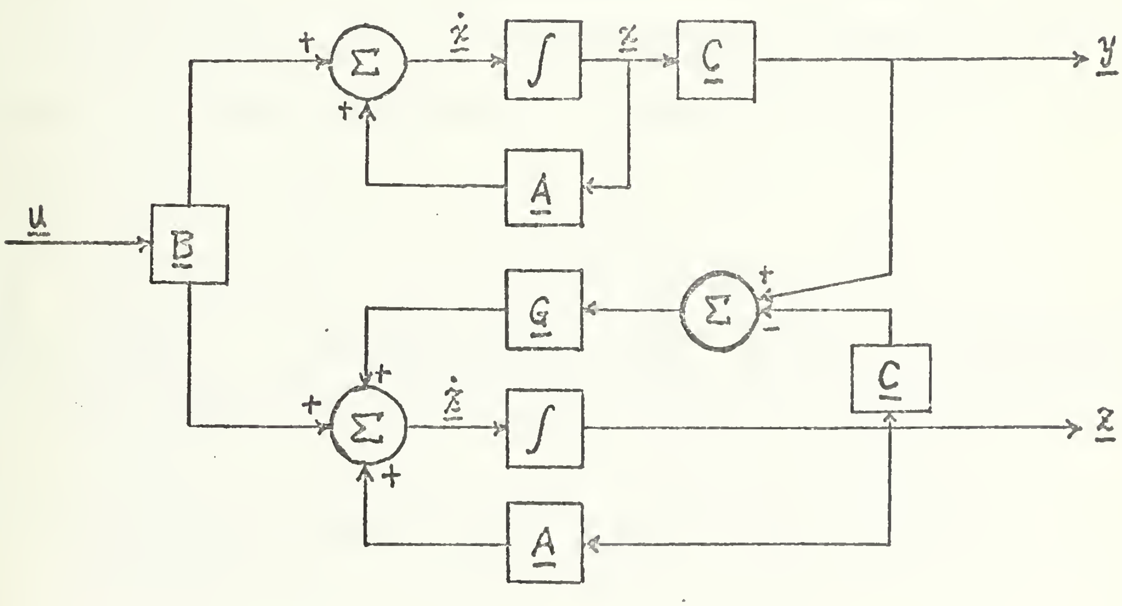


FIGURE 1. FULL-ORDER OBSERVER

C. REDUCED-ORDER OBSERVERS

The full-order observer possesses a certain degree of redundancy. The reason for this redundancy is that the full-order observer constructs an estimate of the entire plant state, but the output of the plant, which is made up of linear combinations of the states, is available by direct measurement. Hence, it is actually unnecessary to construct a system like the full-order observer to estimate the entire plant state.

The redundancy can be eliminated by reducing the dynamic order of the observer system to (n - m), the difference between the dimensions of the state vector $\underline{x}(t)$ and the output vector $\underline{y}(t)$. The observer which is reduced to its minimal dynamic order (n - m) is called a "reduced-order observer." The structure of a reduced-order observer is illustrated in Fig. 2.

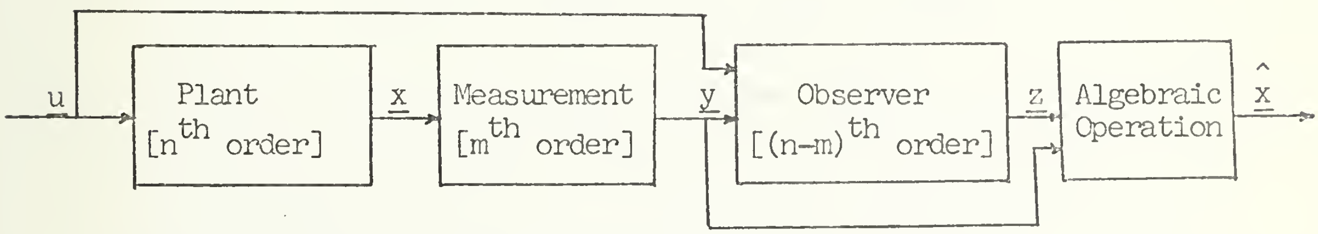


FIGURE 2. STRUCTURE OF REDUCED-ORDER OBSERVER

The starting point for consideration of reduced-order observers is again the plant characterized by

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{u}(t) \tag{12}$$

$$\underline{y}(t) = \underline{C}\ \underline{x}(t) \tag{13}$$

The output distribution matrix $\underline{C}$ has dimension m. By introducing a change of coordinates, it can be assumed that the matrix $\underline{C}$ takes the form $\underline{C} = [\ \underline{I} \ \vdots \ \underline{0}\ ]$. The new state vector $\underline{\tilde{x}}(t)$, is selected so that its first m components are equal to the plant output vector $\underline{y}(t)$ and the remaining (n - m) components, denoted by $\underline{w}(t)$, can be selected arbitrarily but in such a way that the states $\underline{y}(t)$ and $\underline{w}(t)$ are linearly independent. Thus, the new state vector $\underline{\tilde{x}}(t)$ can be written in partitioned form as

$$\underline{\tilde{x}}(t) = \left[\begin{array}{c} \underline{y}(t) \\ \hline \underline{w}(t) \end{array}\right]$$

and $\underline{\tilde{x}}(t)$ is related to the original plant state vector $\underline{x}(t)$ by a nonsingular matrix $\underline{P}$, i.e.,

$$\underline{\tilde{x}}(t) = \underline{P}\ \underline{x}(t) = \left[\begin{array}{c} \underline{C} \\ \hline \underline{D} \end{array}\right] \underline{x}(t)$$

The matrix $\underline{C}$ here is the one given in (13) and $\underline{D}$ is an arbitrary matrix selected in such a way that $\underline{P}$ is nonsingular. The new state equations have the form

$$\left[\begin{array}{c} \dot{\underline{y}}(t) \\ \hline \dot{\underline{w}}(t) \end{array}\right] = \underline{A}' \left[\begin{array}{c} \underline{y}(t) \\ \hline \underline{w}(t) \end{array}\right] + \underline{B}'\ \underline{u}(t) \qquad (14)$$

The matrices $\underline{A}'$ and $\underline{B}'$ in (14) are not the same as $\underline{A}$ and $\underline{B}$ in (12). These two sets of matrices are related to one

another through $\underline{P}$. Equation (14) can be rewritten in partitioned matrix form as

$$\begin{bmatrix} \dot{\underline{y}}(t) \\ \hdashline \dot{\underline{w}}(t) \end{bmatrix} = \left[\begin{array}{c:c} \underline{A}_{11}' & \underline{A}_{12}' \\ \hdashline \underline{A}_{21}' & \underline{A}_{22}' \end{array}\right] \begin{bmatrix} \underline{y}(t) \\ \hdashline \underline{w}(t) \end{bmatrix} + \begin{bmatrix} \underline{B}_1' \\ \hdashline \underline{B}_2' \end{bmatrix} \underline{u}(t) \quad (15)$$

or

$$\dot{\underline{y}}(t) = \underline{A}_{11}' \, \underline{y}(t) + \underline{A}_{12}' \, \underline{w}(t) + \underline{B}_1' \, \underline{u}(t) \quad (16)$$

$$\dot{\underline{w}}(t) = \underline{A}_{21}' \, \underline{y}(t) + \underline{A}_{22}' \, \underline{w}(t) + \underline{B}_2' \, \underline{u}(t) \quad (17)$$

The vector $\underline{y}(t)$ is available for measurement, and if it is differentiated, $\dot{\underline{y}}(t)$ can also be determined. Since $\underline{u}(t)$ is also available, (16) provides $\underline{A}_{12}' \, \underline{w}(t)$ which plays the role of a set of measurement equations for the system (17) which has $\underline{w}(t)$ as its state vector and $\underline{A}_{21}' \, \underline{y}(t) + \underline{B}_2' \, \underline{u}(t)$ as its input vector. The next step is to construct a full-order observer of order (n - m) for the plant of equation (17) using

$$\underline{A}_{12}' \, \underline{w}(t) = \dot{\underline{y}}(t) - \underline{A}_{11}' \, \underline{y}(t) - \underline{B}_1' \, \underline{u}(t)$$

as the measurement. Thus, the observer for the system described by (16) and (17) can be expressed as

$$\dot{\hat{\underline{w}}}(t) = (\underline{A}_{22}' - \underline{G}\,\underline{A}_{12}')\,\hat{\underline{w}}(t) + \underline{A}_{21}'\underline{y}(t) + \underline{B}_{2}'\,\underline{u}(t)$$

$$+ \underline{G}\,[\dot{\underline{y}}\,(t) - \underline{A}_{11}'\,\underline{y}(t)] - \underline{G}\,\underline{B}_{1}'\,\underline{u}(t) \qquad (18)$$

or

$$\dot{\hat{\underline{w}}}(t) = (\underline{A}_{22}' - \underline{G}\,\underline{A}_{12}')\,\hat{\underline{w}} + (\underline{A}_{21}' - \underline{G}\,\underline{A}_{11}')\,\underline{y}(t)$$

$$+ (\underline{B}_{2}' - \underline{G}\,\underline{B}_{1}')\,\underline{u}(t) + \underline{G}\,\dot{\underline{y}}(t) \qquad (19)$$

where $\hat{\underline{w}}$ represents the estimate of $\underline{w}$. The gain matrix $\underline{G}$ in (18) or (19) can be selected so that the observer system coefficient matrix $\underline{A}_{22}' - \underline{G}\,\underline{A}_{12}'$ has arbitrary eigenvalues provided that $(\underline{A}_{12}', \underline{A}_{22}')$ is observable. It is known [4] that if $(\underline{C}, \underline{A})$ is observable, then $(\underline{A}_{12}', \underline{A}_{22}')$ is also observable. The configuration of the reduced-order observer is shown in Fig. 3.

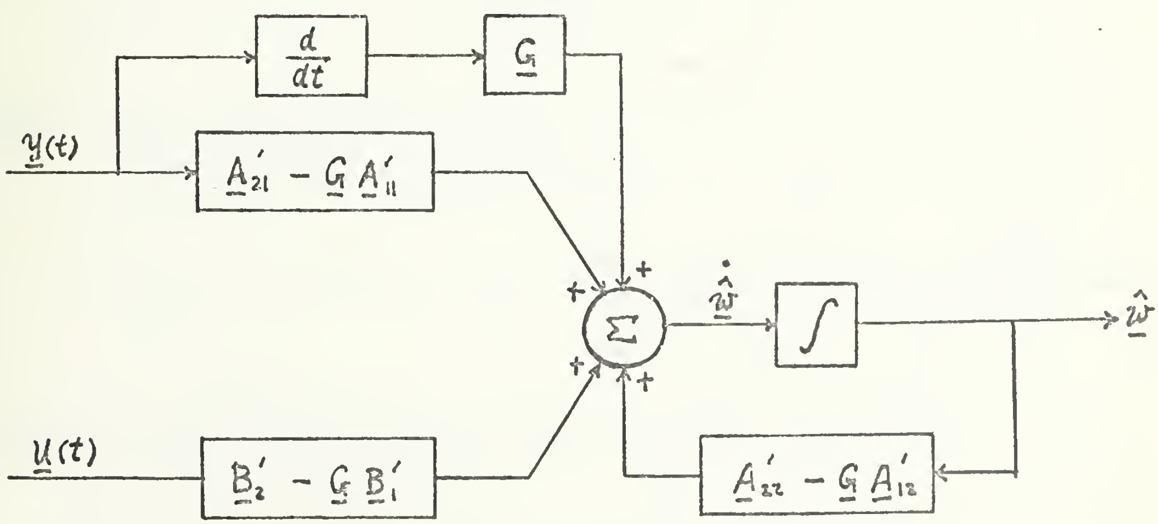


FIGURE 3. REDUCED-ORDER OBSERVER WITH DERIVATIVE

The differentiation of $\underline{y}(t)$ in (19) may be troublesome to perform physically; however, it can be avoided by modifying the observer state equation (19). The resulting modified observer state equation can be written as

$$\dot{\underline{z}}(t) = (\underline{A}_{22}' - \underline{G}\,\underline{A}_{12}')\,\underline{z}(t) + (\underline{A}_{22}' - \underline{G}\,\underline{A}_{12}')\,\underline{G}\,\underline{y}(t)$$

$$+ (\underline{A}_{21}' - \underline{G}\,\underline{A}_{11}')\,\underline{y}(t) + (\underline{B}_2' - \underline{G}\,\underline{B}_1')\,\underline{u}(t) \qquad (20)$$

with

$$\hat{\underline{w}} = \underline{z}(t) + \underline{G}\,\underline{y}(t) \qquad (21)$$

and the corresponding observer diagram is shown in Fig. 4 which is equivalent to the observer in Fig. 3 at the point $\hat{\underline{w}}$.

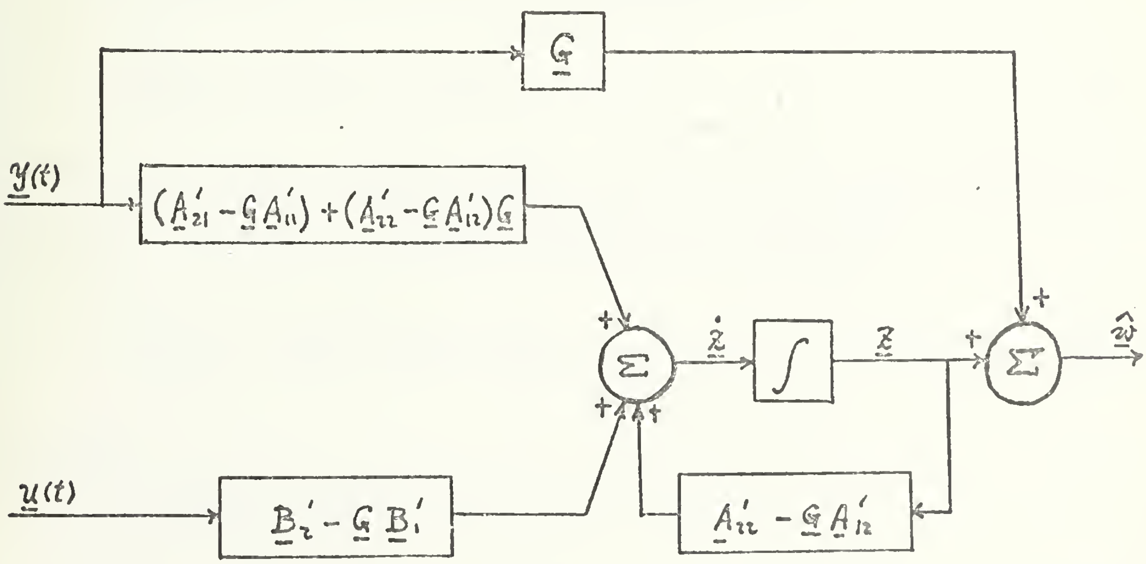


FIGURE 4. MODIFIED REDUCED-ORDER OBSERVER

D. THE CONVENTIONAL METHOD OF OBSERVER DESIGN

It has previously been shown that the full-order observer for the linear time-invariant plant

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{u}(t) \\ \underline{y}(t) &= \underline{C}\ \underline{x}(t) \end{aligned} \tag{22}$$

is characterized by

$$\dot{\underline{z}}(t) = (\underline{A} - \underline{G}\ \underline{C})\ \underline{z}(t) + \underline{G}\ \underline{y}(t) + \underline{B}\ \underline{u}(t) \tag{23}$$

If ($\underline{C}$, $\underline{A}$) is observable, the observer's eigenvalue can be located arbitrarily by selecting the gain matrix $\underline{G}$. The problem is how to select the eigenvalue locations for the observer.

The observer error $\underline{e}(t)$ is defined as

$$\underline{e}(t) \triangleq \underline{z}(t) - \underline{x}(t) \tag{24}$$

Differentiating equation (24), substituting the expressions for $\dot{\underline{z}}(t)$ and $\dot{\underline{x}}(t)$, and solving for $\dot{\underline{e}}(t)$ gives

$$\dot{\underline{e}}(t) = (\underline{A} - \underline{G}\ \underline{C})\ \underline{e}(t) \tag{25}$$

Letting

$$\underline{F} \triangleq \underline{A} - \underline{G}\ \underline{C}$$

then the error equation for the observer can be written as

$$\dot{\underline{e}}(t) = \underline{F}\ \underline{e}(t) \tag{26}$$

The solution to this equation is

$$\underline{e}(t) = \varepsilon^{\underline{F}\, t}\, \underline{e}(0) \tag{27}$$

Thus, the dynamic response of the error between the observer and the plant state is determined by the matrix $\underline{F} = \underline{A} - \underline{G}\,\underline{C}$. It is desired that the error $\underline{e}(t)$ eventually approach zero for all $\underline{e}(0)$. Thus, if $\underline{G}$ is selected so that $\underline{F} = \underline{A} - \underline{G}\,\underline{C}$ is a stability matrix then the observer state $\underline{z}(t)$ approaches the plant state $\underline{x}(t)$ at a rate controlled by $\underline{G}$. One approach to observer design is to select the desired eigenvalues and then determine $\underline{G}$ to obtain these eigenvalues [4].

III. A NEW METHOD FOR OBSERVER DESIGN

A. PROBLEM DESCRIPTION

An alternative procedure for observer design can be based on a performance measure for the error system (26) consisting of the integral of a quadratic form in the observer error, that is,

$$J = \int_{t_o}^{t_f} \underline{e}^T(\tau)\ \underline{Q}\ \underline{e}(\tau)\ d\tau \tag{28}$$

where $\underline{Q}$ is a real symmetric positive definite constant matrix and $\underline{e}(\tau)$ is a function of the gain matrix $\underline{G}$, which is to be selected to minimize this performance measure. It is shown in the Appendix that equation (28) can also be expressed as

$$J = \underline{e}_o^T\ \underline{V}_o(\underline{G})\ \underline{e}_o \tag{29}$$

where $\underline{e}_o$ is the initial error of the observer, $\underline{e}_o^T$ is the transpose of $\underline{e}_o$, and $\underline{V}_o(\underline{G})$ is the value at $\tau = 0$ of the real symmetric positive definite solution $\underline{V}(\tau)$ of the matrix differential equation

$$\dot{\underline{V}}(\tau) = -\underline{F}^T\ \underline{V}(\tau) - \underline{V}(\tau)\ \underline{F} - \underline{Q} \tag{30}$$

subject to the boundary condition

$$\underline{V}(t_f) = \underline{0} \tag{31}$$

Alternatively, defining $t = t_f - \tau$, this equation can be expressed as

$$\dot{\underline{V}}(t) = \underline{F}^T \underline{V}(t) + \underline{V}(t) \underline{F} + \underline{Q} \tag{32}$$

with the initial condition

$$\underline{V}(0) = \underline{0} \tag{33}$$

From equation (29) it is seen that J depends on the initial error of the observer, $\underline{e}_o$, and on the matrix $\underline{V}_o(\underline{G})$ which is implicitly dependent on $\underline{G}$ through equation (30). Finding a solution to equation (30) requires solving $n(n + 1)/2$ differential equations (since $\underline{V}$ is a symmetric matrix).

B. CONSIDERATION OF THE INITIAL CONDITION

It is desired to select the observer gain matrix $\underline{G}$ to minimize the performance measure

$$J = \int_{t_o}^{t_f} \underline{e}^T(\tau)\, \underline{Q}\, \underline{e}(\tau)\, d\tau = \underline{e}_o^T\, \underline{V}_o(\underline{G})\, \underline{e}_o$$

To calculate the performance measure, a priori knowledge of the initial state of the error system is needed. The initial error state is not known, but a reasonable approach is to take the worst initial state in some bounded region. It will be assumed that the initial observer state is $\underline{0}$ and that the possible initial error states lie within a hyperspherical

region with center at the origin and having a specified radius ρ. With this assumption the performance measure can be written as

$$J_\rho = \max_{\substack{\underline{e}_o \\ \|\underline{e}_o\| \leq \rho}} \left\{ \int_{t_o}^{t_f} \underline{e}^T(\tau)\, \underline{Q}\, \underline{e}(\tau)\, d\tau \right\}$$

$$= \max_{\substack{\underline{e}_o \\ \|\underline{e}_o\| \leq \rho}} \{\underline{e}_o^T\, \underline{V}_o(\underline{G})\, \underline{e}_o\} \tag{34}$$

where $\| \ \|$ denotes the Euclidean norm. It is easily shown that the maximum occurs where $\|\underline{e}_o\| = \rho$, moreover, without loss of generality it can be assumed that $\rho = 1$. That is, the maximum can be considered to occur on the unit hypersphere. This is because, for any $\underline{e}_o$ such that $\|\underline{e}_o\| = 1$ and any radius $\rho > 0$,

$$J_\rho = \rho\, \underline{e}_o^T\, \underline{V}_o(\underline{G})\, \rho\, \underline{e}_o = \rho^2\, \underline{e}_o^T\, \underline{V}_o(\underline{G})\, \underline{e}_o$$

$$= \rho^2 J_1 \tag{35}$$

i.e., the maximum performance measure on the hypersphere of radius ρ is ρ^2 times the maximum performance measure on the unit hypersphere. Hence, it is desired to minimize

$$J_1 = \max_{\substack{\underline{e}_o \\ \|\underline{e}_o\| = 1}} \left\{ \int_{t_o}^{t_f} \underline{e}^T(\tau)\ \underline{Q}\ \underline{e}(\tau)\ d\tau \right\}$$

$$= \max_{\substack{\underline{e}_o \\ \|\underline{e}_o\| = 1}} \left\{ \underline{e}_o^T\ \underline{V}_o(G)\ \underline{e}_o \right\} \tag{36}$$

with respect to $\underline{G}$.

C. EXTENSION TO REDUCED-ORDER OBSERVERS

So far, a new method to construct a full-order observers has been discussed. The method can easily be applied to design reduced-order observers also. Consider again the linear time-invariant system

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{u}(t) \tag{37}$$

$$\underline{y}(t) = \underline{C}\ \underline{x}(t) \tag{38}$$

The reduced-order observer equation for this system is

$$\begin{aligned} \dot{\underline{z}}(t) = {} & (\underline{A}_{22}' - \underline{G}\ \underline{A}_{12}')\ \underline{z}(t) + (\underline{A}_{22}' - \underline{G}\ \underline{A}_{12}')\ \underline{G}\ \underline{y}(t) \\ & + (\underline{A}_{21}' - \underline{G}\ \underline{A}_{11}')\ \underline{y}(t) + (\underline{B}_2' - \underline{G}\ \underline{B}_1')\ \underline{u}(t) \end{aligned} \tag{39}$$

with

$$\hat{\underline{w}}(t) = \underline{z}(t) + \underline{G}\,\underline{y}(t) \tag{40}$$

where $\hat{\underline{w}}(t)$ is the estimate of $\underline{w}(t)$.

The characteristics of the observer system (39) depend on the coefficient matrix $\underline{A}_{22}' - \underline{G}\,\underline{A}_{12}'$ which depends on the gain matrix $\underline{G}$. Again to determine the gain matrix $\underline{G}$, consider the state error, $\hat{\underline{w}}(t) - \underline{w}(t)$. The observer error $\underline{e}(t)$ is

$$\underline{e}(t) \triangleq \hat{\underline{w}}(t) - \underline{w}(t) \tag{41}$$

Differentiating equation (41) gives

$$\dot{\underline{e}}(t) = \dot{\hat{\underline{w}}}(t) - \dot{\underline{w}}(t)$$

Substituting $\dot{\hat{\underline{w}}}(t)$ and $\dot{\underline{w}}(t)$ and solving for $\dot{\underline{e}}(t)$ yields

$$\underline{e}(t) = (\underline{A}_{22}' - \underline{G}\,\underline{A}_{12}')\,\underline{e}(t) \tag{42}$$

Letting $\underline{F} \triangleq \underline{A}_{22}' - \underline{G}\,\underline{A}_{12}'$, then (42) can be written as

$$\dot{\underline{e}}(t) = \underline{F}\,\underline{e}(t) \tag{43}$$

The solution to this equation is

$$\underline{e}(t) = \varepsilon^{\underline{F}\,t}\,\underline{e}(0) \tag{44}$$

Notice that equation (44) has exactly the same form as equation (27) except for the definition of the matrix $\underline{F}$. Therefore, the results obtained for full-order observers apply to reduced-order observers as well.

D. SIMPLIFICATION FOR THE INFINITE-TIME CASE

It will now be assumed that the upper limit on the performance measure integral is infinity. It will be shown that if $t_f \to \infty$, the problem of calculating $\underline{V}_o(\underline{G})$ can be simplified to the problem of solving a set of $n(n+1)/2$ linear algebraic equations. Let us state an important lemma and two theorems related to this simplification. The first step is to write an explicit solution for $\underline{V}(t)$.

Lemma 1: The solution of the matrix differential equation

$$\dot{\underline{V}}(t) = \underline{F}^T \underline{V}(t) + \underline{V}(t) \underline{F} + \underline{Q} \tag{45}$$

is

$$\underline{V}(t) = \underline{\phi}(t) \underline{V}(0) \underline{\phi}^T(t) + \int_0^t \underline{\phi}(t-\tau) \underline{Q} \underline{\phi}^T(t-\tau)\, d\tau \tag{46}$$

where $\underline{\phi}(t)$ is the transition matrix for the system

$$\dot{\underline{z}}(t) = \underline{F}^T \underline{z}(t) \tag{47}$$

Proof: Suppose that (46) is the solution of (45), and is rewritten as

$$\underline{V}(t) = \underline{\phi}(t) \underline{V}(0) \underline{\phi}^T(t) + \underline{\phi}(t) \left[\int_0^t \underline{\phi}(-\tau) \underline{Q} \underline{\phi}^T(-\tau)\, d\tau \right] \underline{\phi}^T(t)$$

$$\triangleq \underline{\phi}(t) \underline{V}(0) \underline{\phi}^T(t) + \underline{\phi}(t) \underline{L}(t) \underline{\phi}^T(t) \tag{48}$$

where $\underline{L}(t)$ is defined as

$$\underline{L}(t) \triangleq \int_0^t \underline{\phi}(-\tau) \underline{Q} \underline{\phi}^T(-\tau)\, d\tau \tag{49}$$

Differentiating (48) gives

$$\dot{\underline{V}}(t) = \dot{\underline{\phi}}(t)\ \underline{V}(0)\ \underline{\phi}^T(t) + \underline{\phi}(t)\ \underline{V}(0)\ \dot{\underline{\phi}}^T(t) + \dot{\underline{\phi}}(t)\ \underline{L}(t)\ \underline{\phi}^T(t)$$

$$+ \underline{\phi}(t)\ \underline{L}(t)\ \dot{\underline{\phi}}^T(t) + \underline{\phi}(t)\ [\underline{\phi}(-t)\ \underline{Q}\ \underline{\phi}^T(-t)]\ \underline{\phi}^T(t) \tag{50}$$

The last term of equation (50) becomes $\underline{Q}$, because

$$\underline{\phi}(t)\ [\underline{\phi}(-t)\ \underline{Q}\ \underline{\phi}^T(-t)]\ \underline{\phi}^T(t) = \underline{\phi}(0)\ \underline{Q}\ \underline{\phi}^T(0) = \underline{I}\ \underline{Q}\ \underline{I} = \underline{Q}$$

Also

$$\dot{\underline{\phi}}(t) = \underline{F}^T\ \underline{\phi}(t) \tag{51}$$

therefore (50) can be written as

$$\dot{\underline{V}}(t) = \underline{F}^T\ \underline{\phi}(t)\ \underline{V}(0)\ \underline{\phi}^T(t) + \underline{\phi}(t)\ \underline{V}(0)\ \underline{\phi}^T(t)\ \underline{F}$$

$$+ \underline{F}^T\ \underline{\phi}(t)\ \underline{L}(t)\ \underline{\phi}^T(t) + \underline{\phi}(t)\ \underline{L}(t)\ \underline{\phi}^T(t)\ \underline{F} + \underline{Q} \tag{52}$$

But from equation (48)

$$\underline{\phi}(t)\ \underline{V}(0)\ \underline{\phi}^T(t) = \underline{V}(t) - \underline{\phi}(t)\ \underline{L}(t)\ \underline{\phi}^T(t) \tag{53}$$

Substituting (53) into (52) gives

$$\dot{\underline{V}}(t) = \underline{F}^T\ [\underline{V}(t) - \underline{\phi}(t)\ \underline{L}(t)\ \underline{\phi}^T(t)] + [\underline{V}(t) - \underline{\phi}(t)\ \underline{L}(t)\ \underline{\phi}^T(t)]\ \underline{F}$$

$$+ \underline{F}^T\ \underline{\phi}(t)\ \underline{L}(t)\ \underline{\phi}^T(t) + \underline{\phi}(t)\ \underline{L}(t)\ \underline{\phi}^T(t)\ \underline{F} + \underline{Q} \tag{54}$$

Simplifying equation (54) yields the final desired result

$$\dot{\underline{V}}(t) = \underline{F}^T\ \underline{V}(t) + \underline{V}\ \underline{F} + \underline{Q}$$

The following well-known theorem gives a necessary and sufficient condition for the error system (26) to be asymptotically stable, that is,

$$\lim_{t\to\infty} \underline{e}(t) = \underline{0} \qquad \text{for all } \underline{e}(0).$$

Theorem 1: The linear time-invariant error system

$$\dot{\underline{e}}(t) = \underline{F}\ \underline{e}(t) \tag{55}$$

is asymptotically stable if and only if, given any positive definite symmetric matrix $\underline{Q}$, there exists a positive definite symmetric matrix $\underline{K}$ such that

$$\underline{F}^T \underline{K} + \underline{K}\ \underline{F} = -\ \underline{Q} \tag{56}$$

Theorem 1 has been proved in [2]. This theorem can be used to determine whether or not the error system (26) is asymptotically stable for any choice of the gain matrix $\underline{G}$.

The following theorem applies specifically to the case where $t_f \to \infty$.

Theorem 2: Assume that the linear time-invariant system

$$\dot{\underline{e}}(t) = \underline{F}\ \underline{e}(t) \tag{57}$$

is asymptotically stable, and has a performance measure defined by

$$J = \int_0^{t_f} \underline{e}^T(\tau)\ \underline{Q}\ \underline{e}(\tau)\ d\tau \tag{58}$$

which is equivalent to

$$J = \underline{e}_o^T \, \underline{V}_o(\underline{G}) \, \underline{e}_o \tag{59}$$

where $\underline{V}(t)$ satisfies

$$\dot{\underline{V}}(t) = \underline{F}^T \underline{V}(t) + \underline{V}(t) \underline{F} + \underline{Q}$$

with

$$\underline{V}(0) = \underline{0}$$

then

$$\dot{\underline{V}}(t) = \underline{F}^T \underline{V}(t) + \underline{V}(t) \underline{F} + \underline{Q} \tag{60}$$

has a constant steady-state solution $\underline{V}_o$, i.e.,

$$\lim_{t \to \infty} \underline{V}(t) = \underline{V}_o \quad \text{(a constant)}$$

Proof: By Lemma 1 with $\underline{V}(0) = \underline{0}$, the solution to (60) is

$$\underline{V}(t) = \int_0^t \underline{\phi}(t - \tau) \, \underline{Q} \, \underline{\phi}^T(t - \tau) \, d\tau \tag{61}$$

Substituting $\underline{\phi}(t) = \varepsilon^{\underline{F}^T t}$ and $\underline{\phi}^T(t) = \varepsilon^{\underline{F} t}$ into (61) and using properties of the matrix exponential, gives

$$\underline{V}(t) = \varepsilon^{\underline{F}^T t} \int_0^t \varepsilon^{-\underline{F}^T \tau} \, \underline{Q} \, (\varepsilon^{\underline{F}^T t} \varepsilon^{-\underline{F}^T \tau})^T \, d\tau$$

$$= \varepsilon^{\underline{F}^T t} \int_0^t \varepsilon^{-\underline{F}^T \tau} \, \underline{Q} \, \varepsilon^{-\underline{F} \tau} \, d\tau \, \varepsilon^{\underline{F}^T} \tag{62}$$

But the system (57) is asymptotically stable, so according to Theorem 1 there exists a positive definite symmetric matrix $\underline{V}_o$ such that

$$\underline{Q} = - \underline{F}^T \underline{V}_o - \underline{V}_o \underline{F} \tag{63}$$

Substituting (63) into (62) gives

$$\underline{V}(t) = - \varepsilon^{\underline{F}^T t} \int_0^t \varepsilon^{-\underline{F}^T \tau} [\underline{F}^T \underline{V}_o + \underline{V}_o \underline{F}] \varepsilon^{-\underline{F}\tau} d\tau \, \varepsilon^{\underline{F}t} \tag{64}$$

Since $\varepsilon^{-\underline{F}^T \tau}$ and $\underline{F}^T$ commute, this can be written as

$$\underline{V}(t) = -\varepsilon^{\underline{F}^T t} \int_0^t - \frac{d}{d\tau} [\varepsilon^{-\underline{F}^T \tau} \underline{V}_o \varepsilon^{-\underline{F}\tau}] \, \varepsilon^{\underline{F}t}$$

$$= \varepsilon^{\underline{F}^T t} [\varepsilon^{-\underline{F}^T \tau} \underline{V}_o \varepsilon^{-\underline{F}\tau} \Big|_0^t] \, \varepsilon^{\underline{F}t}$$

$$= \varepsilon^{\underline{F}^T t} \varepsilon^{-\underline{F}^T t} \underline{V}_o \varepsilon^{-\underline{F}t} \varepsilon^{\underline{F}t} - \varepsilon^{\underline{F}^T t} \underline{I} \, \underline{V}_o \underline{I} \, \varepsilon^{\underline{F}t}$$

$$= \underline{I} \, \underline{V}_o \underline{I} - \varepsilon^{\underline{F}^T t} \underline{V}_o \varepsilon^{\underline{F}t} \tag{65}$$

The system (57) is stable, so

$$\lim_{t\to\infty} \varepsilon^{\underline{F}t} = \underline{0} \quad \text{and} \quad \lim_{t\to\infty} \varepsilon^{\underline{F}^T t} = \underline{0} \tag{66}$$

Hence, substituting (66) into (65)

$$\lim_{t\to\infty} \underline{V}(t) = \underline{V}_o \quad \text{(a constant)}$$

which proves the theorem.

According to the result of Theorem 2, $\dot{\underline{V}}(t)$ becomes zero as $t \to \infty$. Therefore, $\underline{V}_o$ is the solution of

$$\underline{F}^T \underline{V} + \underline{V} \underline{F} = - \underline{Q} \tag{67}$$

which is obtained by setting $\dot{\underline{V}}(t) = \underline{0}$ in equation (32). Equation (67) gives a set of $n(n+1)/2$ linear algebraic equations that can be solved for $\underline{V}$ after any positive definite $\underline{Q}$ has been selected for the performance measure (28). Hence, the problem of calculating $\underline{V}_o$ becomes simple. The positive definiteness of the solution $\underline{V}$ can be checked by using Sylvester's criterion [7] and if and only if $\underline{V}$ is positive definite the system is asymptotically stable.

IV. SOLUTION OF THE OBSERVER DESIGN PROBLEM

As mentioned before, the error equation needed to determine the gain matrix $\underline{G}$ has the same form for both the full-order and reduced-order observers. So one design method can be applied for both cases.

A. DEVELOPMENT OF THE METHOD

It is desired that the gain matrix $\underline{G}$ be found which minimizes

$$J_1 = \max_{\substack{\underline{e}_o \\ \|\underline{e}_o\| = 1}} \left\{ \int_0^\infty \underline{e}^T(\tau)\ \underline{Q}\ \underline{e}(\tau)\ d\tau \right\}$$

$$= \max_{\substack{\underline{e}_o \\ \|\underline{e}_o\| = 1}} \left[\underline{e}_o^T\ \underline{V}_o(\underline{G})\ \underline{e}_o \right] \qquad (68)$$

Letting min $J_1 = J_1{}^*$, then

$$J_1{}^* = \min_{\underline{G}}\ \max_{\substack{\underline{e}_o \\ \|\underline{e}_0\| = 1}} \left[\underline{e}_o^T\ \underline{V}_o(G)\ \underline{e}_o\right] \qquad (69)$$

This is a minimax problem, and equation (69) is a real quadratic form in $\underline{e}_o$. There is a useful theorem about the extremal properties of the eigenvalues of a real quadratic form [1].

Theorem 3: The global maximum of a real quadratic form on the unit hypersphere is equal to the largest eigenvalue of the quadratic form, and moreover the corresponding eigenvector is the vector drawn from the origin to the point on the hypersphere where the quadratic form achieves its maximum.

Therefore

$$\max_{\substack{\underline{e}_o \\ \|e_o\| = 1}} \left[\underline{e}_o^T \underline{V}_o(\underline{G}) \underline{e}_o \right] = \lambda_1(\underline{G}) \tag{70}$$

where $\lambda_1(\underline{G})$ is the largest eigenvalue of $\underline{V}_o(\underline{G})$. Hence, now the problem is to minimize $\lambda_1(\underline{G})$ with respect to the elements of $\underline{G}$; that is, to find the observer gain matrix $\underline{G}$ such that

$$J_1^* = \min_{\underline{G}} \lambda_1(\underline{G}) \tag{71}$$

To ensure that the observer is stable and does not have an unlimited bandwidth, the gain matrix $\underline{G}$ will be selected from the set $\mathcal{G}$ defined as

$$\mathcal{G} = \left\{ \underline{G} : \begin{array}{ll} 1. & \text{All eigenvalues of } \underline{F} \text{ have negative real parts.} \\ 2. & \text{Sum of eigenvalues of } \underline{F} = \text{trace } \underline{F} \geq -D \end{array} \right\}$$

where D is a specified positive constant. For a given value of $\underline{G}$ the largest eigenvalue is easily found by using a digital computer. The program required is simple because very efficient methods are available for finding the largest eigenvalue of a real symmetric matrix, and many computer facilities have several standard subroutines that can be used for this purpose.

B. ALGORITHM FOR COMPUTING THE GAIN MATRIX

The steps in the computational procedure for finding the optimal gain matrix are given below.

1. Guess $\underline{G}$ subject to the following requirements:

a. $\underline{F}$ is stable. This can be done by assigning arbitrary eigenvalues with negative real parts to $\underline{F}$ and using Luenberger's method [4], for example.

b. Trace $\underline{F} \geq -D$

2. Use a minimization subroutine to minimize

$$J_1 = \max_{\substack{\underline{e}_o \\ \|\underline{e}_o\| = 1}} \left\{ \underline{e}_o^T \, \underline{V}_o(\underline{G}) \, \underline{e}_o \right\} = \lambda_1(\underline{G})$$

Actually, a penalty function is added to ensure that the trace $\underline{F} \geq -D$ constraint is satisfied, e.g.,

$$J_a = J_1 + \begin{cases} 0, & \text{if trace } \underline{F} \geq -D \\ \alpha \times [\text{trace } \underline{F} + D]^2, & \text{if trace } \underline{F} \leq -D \end{cases}$$

$$\triangleq J_1 + J_p$$

where α is a positive weighting factor.

In computing values of J_a for the minimization subroutine the following steps are carried out.

a. Solve $\underline{F}^T \underline{V} + \underline{V} \underline{F} = - \underline{Q}$ for $\underline{V} = \underline{V}_o$

b. Check to see that $\underline{V}_o$ is positive definite. If it is continue, otherwise set $J_a = 10^5$, that is, a large positive number and return to the minimization subroutine.

c. If $\underline{V}_o$ is positive definite it is the steady-state solution of

$$\dot{\underline{V}}(t) = \underline{F}^T \underline{V}(t) + \underline{V}(t) \underline{F} + \underline{Q}$$

Next find the largest eigenvalue of $\underline{V}_o$ using a suitable subroutine.

d. Calculate the trace of $\underline{F}$ and compute the penalty function J_p.

e. Set $J_a = J_1 + J_p$ and return to the minimization subroutine.

In applying this algorithm to the examples given in this thesis the minimizations were accomplished using a subroutine "DIRECT" (NPS Computer Facility) that performs the pattern search method of Hooke and Jeeves, and the largest eigenvalues were found using a subroutine "GIVHO" (NPS Computer Facility) that utilizes the Givens-Householder method.

V. APPLICATIONS OF THE ALGORITHM

A. A FULL-ORDER OBSERVER

The purpose of this example is to illustrate the application of the described algorithm to the design of a full-order observer for a second-order system.

Consider the plant $\frac{Y(S)}{U(S)} = \frac{1}{S^2 + 3S + 2}$ with the state-variable representation

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{u}(t)$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{72}$$

$$\underline{y}(t) = \underline{C}\ \underline{x}(t)$$

$$= [1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{73}$$

A full-order observer for this system is determined by specifying the observer gain matrix

$$\underline{G} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

The observer system coefficient matrix is

$$\underline{F} = \underline{A} - \underline{G}\,\underline{C} = \begin{bmatrix} -g_1 & 1 \\ -2-g_2 & -3 \end{bmatrix} \tag{74}$$

which has the characteristic equation

$$\lambda^2 + (3+g_1)\lambda + 3g_1 + g_2 + 2 = 0 \tag{75}$$

Selecting the observer eigenvalues arbitrarily at $-3 \pm j3$ gives the characteristic equation

$$(\lambda+3+j3)\,(\lambda+3-j3) = \lambda^2 + 6\lambda + 18 = 0 \tag{76}$$

Equating the coefficients of like powers of λ in (75) and (76) yields

$$\underline{G} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

This is the starting guessed value of the gain matrix $\underline{G}$. If one arbitrarily chooses $\underline{Q} = \underline{I}$, and solves (67) for $\underline{V}$ the result is

$$\begin{aligned} V_{11} &= \frac{15 + 3g_1 + 5g_2 + g_2^{\,2}}{12 + 22g_1 + 6g_2 + 2g_1g_2 + 6g_1^{\,2}} \\ V_{12} &= \frac{3 - 2g_1 - g_1g_2}{12 + 22g_1 + 6g_2 + 2g_1g_2 + 6g_1^{\,2}} \\ V_{22} &= \frac{3 + 3g_1 + g_2 + g_1^{\,2}}{12 + 22g_1 + 6g_2 + 2g_1g_2 + 6g_1^{\,2}} \end{aligned} \tag{77}$$

Due to the symmetry of $\underline{V}$ only three equations are needed. If the sum of the observer eigenvalues is to be limited to -10.0, for example, then the proposed algorithm gives the solution for the observer gain matrix as

$$\underline{G} = \begin{bmatrix} 7.0 \\ -0.64063 \end{bmatrix} \tag{78}$$

which gives the observer eigenvalues as

$$\underline{\lambda} = \begin{bmatrix} -6.625 \\ -3.375 \end{bmatrix} \tag{79}$$

For other selections of the weighting matrix $\underline{Q}$ and different limiting values for the trace of $\underline{F}$ the values obtained for the observer gain matrix $\underline{G}$ and the corresponding eigenvalues are tabulated in Table 1. Looking at the table it can be seen that the two eigenvalues become farther apart from each other when the limiting value of the sum of the eigenvalues is decreased (assuming a fixed $\underline{Q}$).

To compare this full-order observer, which will be referred to as the optimal observer, with other observers having different observer eigenvalues, consider two observers having eigenvalues selected as

$$\underline{\lambda} = \begin{bmatrix} -5.0 \\ -5.0 \end{bmatrix} \quad , \quad \underline{\lambda} = \begin{bmatrix} -8.0 \\ -2.0 \end{bmatrix} \tag{80}$$

respectively. The corresponding observer gain matrices are

Weighting Matrix $\underline{Q}$	Sum of Eigenvalues of $\underline{F}$ / Computed $\underline{G}$ and $\underline{\lambda}$	-10	-30	-50	-100
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	g_1	7.0	27.0	47.0	97.0
	g_2	-0.64063	2.4375	5.6875	13.75
	λ_1	-6.625	-26.814	-46.825	-96.832
	λ_2	-3.375	-3.186	-3.175	-3.168
$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$	g_1	7.0	27.0	47.0	97.0
	g_2	-0.53516	0.71875	3.85938	11.875
	λ_1	-6.592	-26.886	-46.867	-96.852
	λ_2	-3.408	-3.114	-3.133	-3.148
$\begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$	g_1	7.0	27.0	47.0	97.0
	g_2	0.44141	-0.95313	1.5625	9.5
	λ_1	-6.2485	-26.9563	-46.919	-96.878
	λ_2	-3.7515	-3.0437	-3.081	-3.122

TABLE 1. OBSERVER GAINS AND THE CORRESPONDING EIGENVALUES

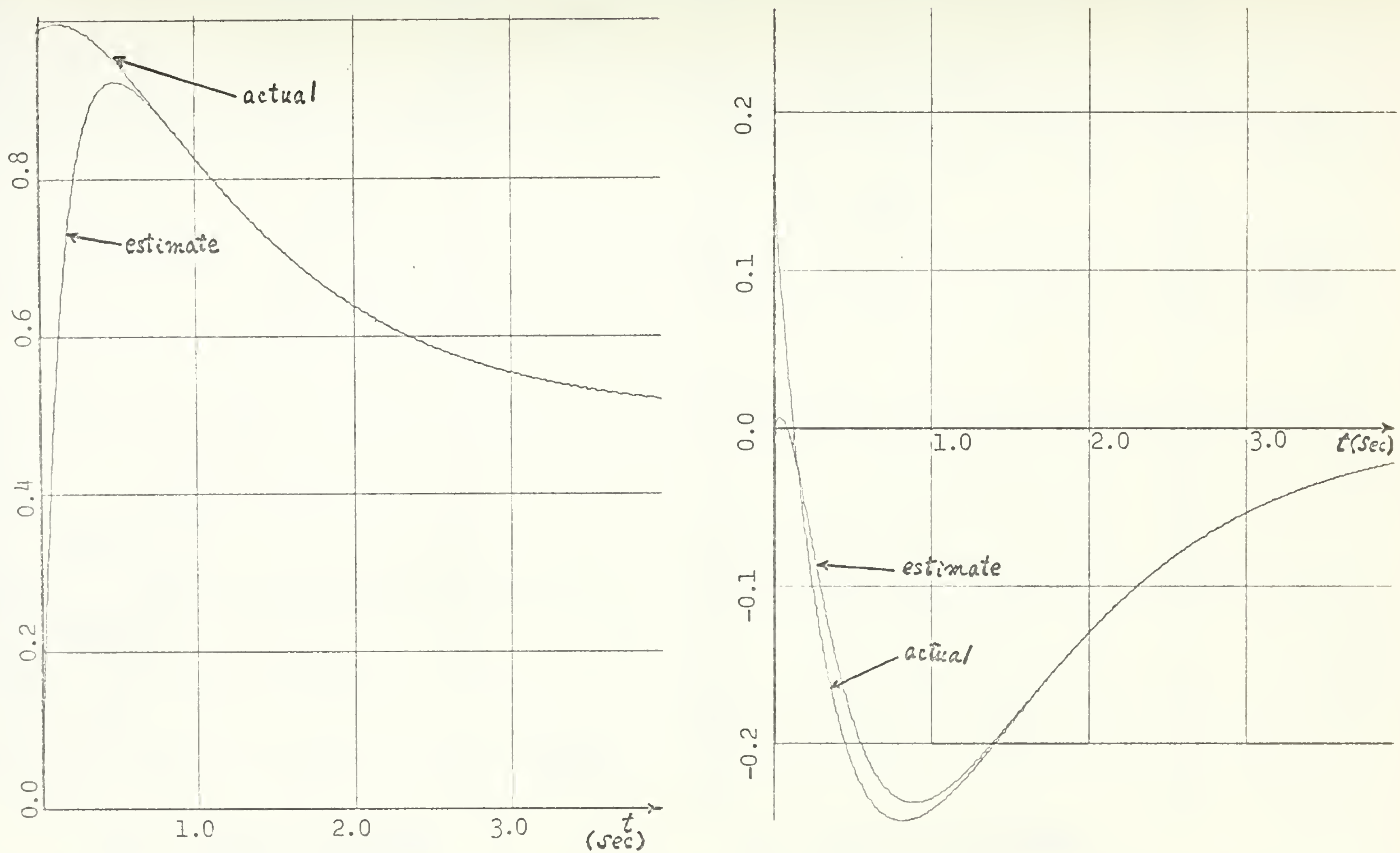
$$\underline{G} = \begin{bmatrix} 7.0 \\ 2.0 \end{bmatrix}, \qquad \underline{G} = \begin{bmatrix} 7.0 \\ -7.0 \end{bmatrix} \tag{81}$$

respectively.

For simulation purposes, the input u(t) was selected to be a unit step function. Assuming that the initial error states lie on the unit circle, the actual and the estimated states computed by using the optimal gains of equation (78) are plotted as functions of time on the graphs in Fig. 5 for the best initial error condition and in Fig. 6 for the worst initial error condition. The estimate errors for the observers with three different observer eigenvalues are plotted on one graph in Figs. 7 and 8 for the best and worst initial error conditions, respectively. The values of the performance measure for the observer error systems with different observer eigenvalues (different observer gains) are tabulated in Table 2 for the worst initial conditions.

Observer Gains (Eigenvalues)	Worst Initial Conditions	Values of Performance Measure
$g_1 = 7.0$, $g_2 = -0.64063$ $\begin{pmatrix} \lambda_1 = -6.625 \\ \lambda_2 = -3.375 \end{pmatrix}$	$\underline{e}(0) = \begin{bmatrix} 0.16 \\ -0.987 \end{bmatrix}$	0.164
$g_1 = 7.0$, $g_2 = 2.0$ $\begin{pmatrix} \lambda_1 = -5.0 \\ \lambda_2 = -5.0 \end{pmatrix}$	$\underline{e}(0) = \begin{bmatrix} 0.525 \\ -0.85 \end{bmatrix}$	0.181
$g_1 = 7.0$, $g_2 = -7.0$ $\begin{pmatrix} \lambda_1 = -8.0 \\ \lambda_2 = -2.0 \end{pmatrix}$	$\underline{e}(0) = \begin{bmatrix} 0.63 \\ 0.777 \end{bmatrix}$	0.303

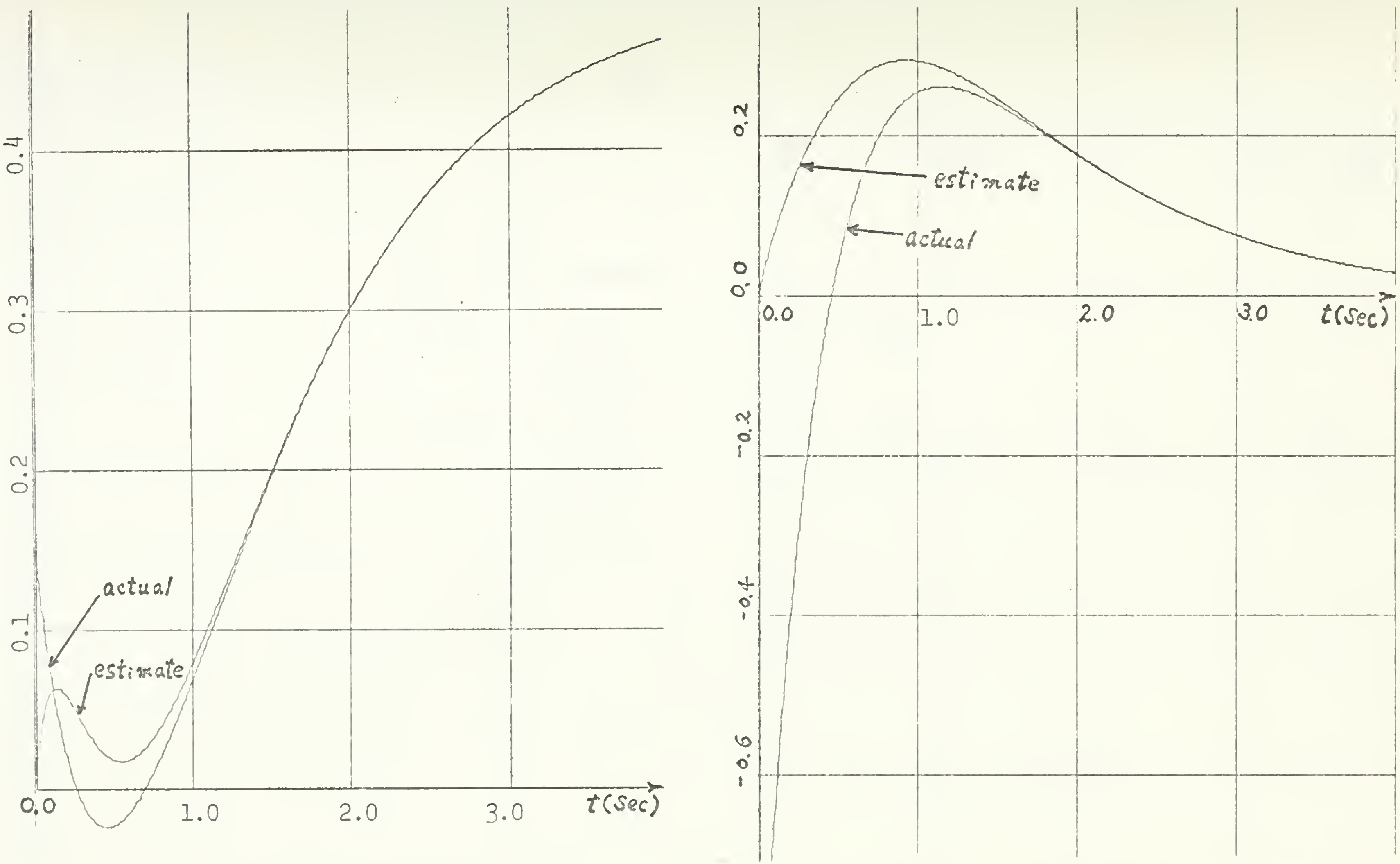
TABLE 2. PERFORMANCE VALUES FOR VARIOUS OBSERVER EIGENVALUES



(a) x(1) AND ITS ESTIMATE

(b) x(2) AND ITS ESTIMATE

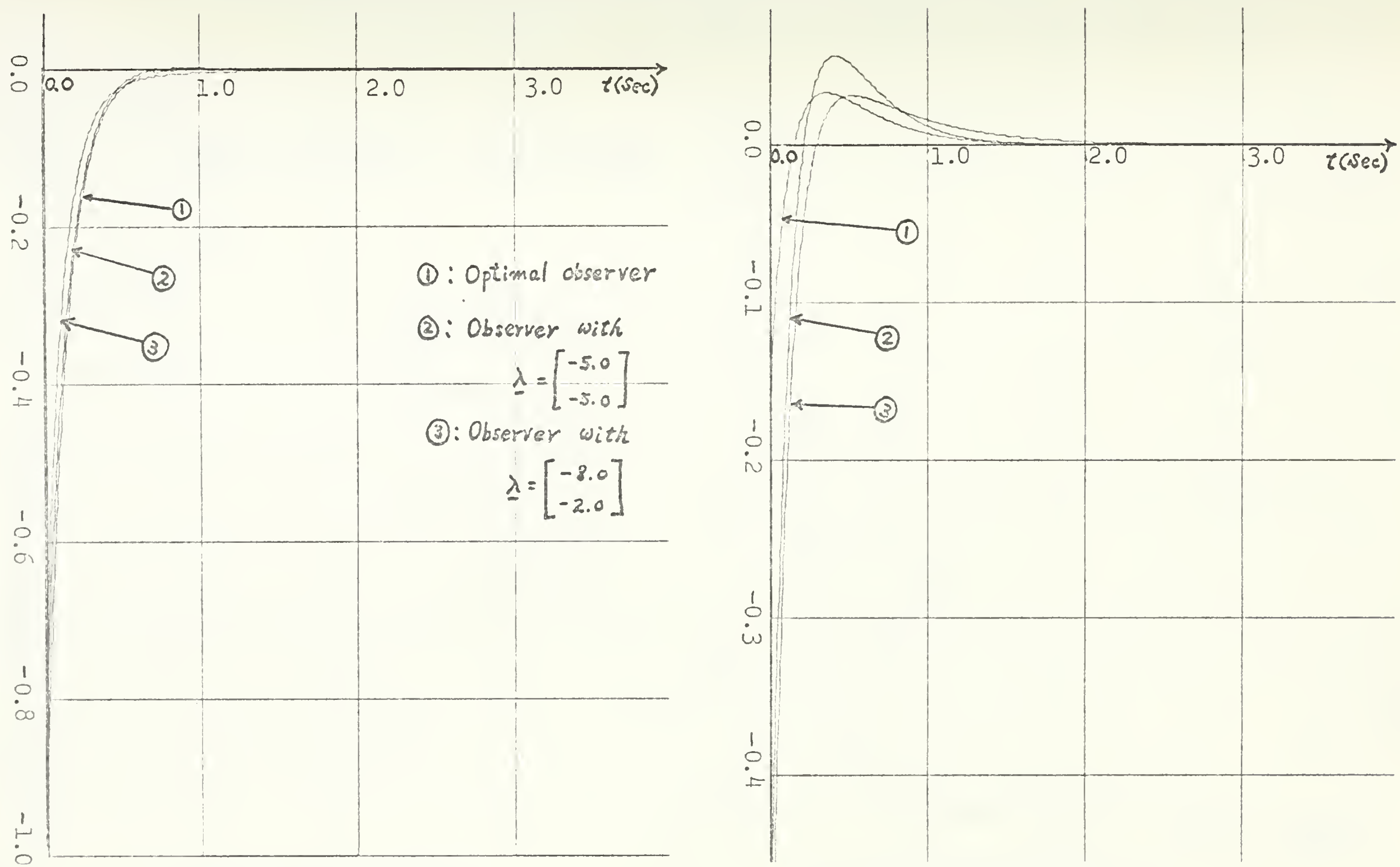
FIGURE 5. ACTUAL STATES AND ESTIMATES USING THE OPTIMAL GAINS FOR THE BEST INITIAL CONDITION



(a) x(1) AND ITS ESTIMATE

(b) x(2) AND ITS ESTIMATE

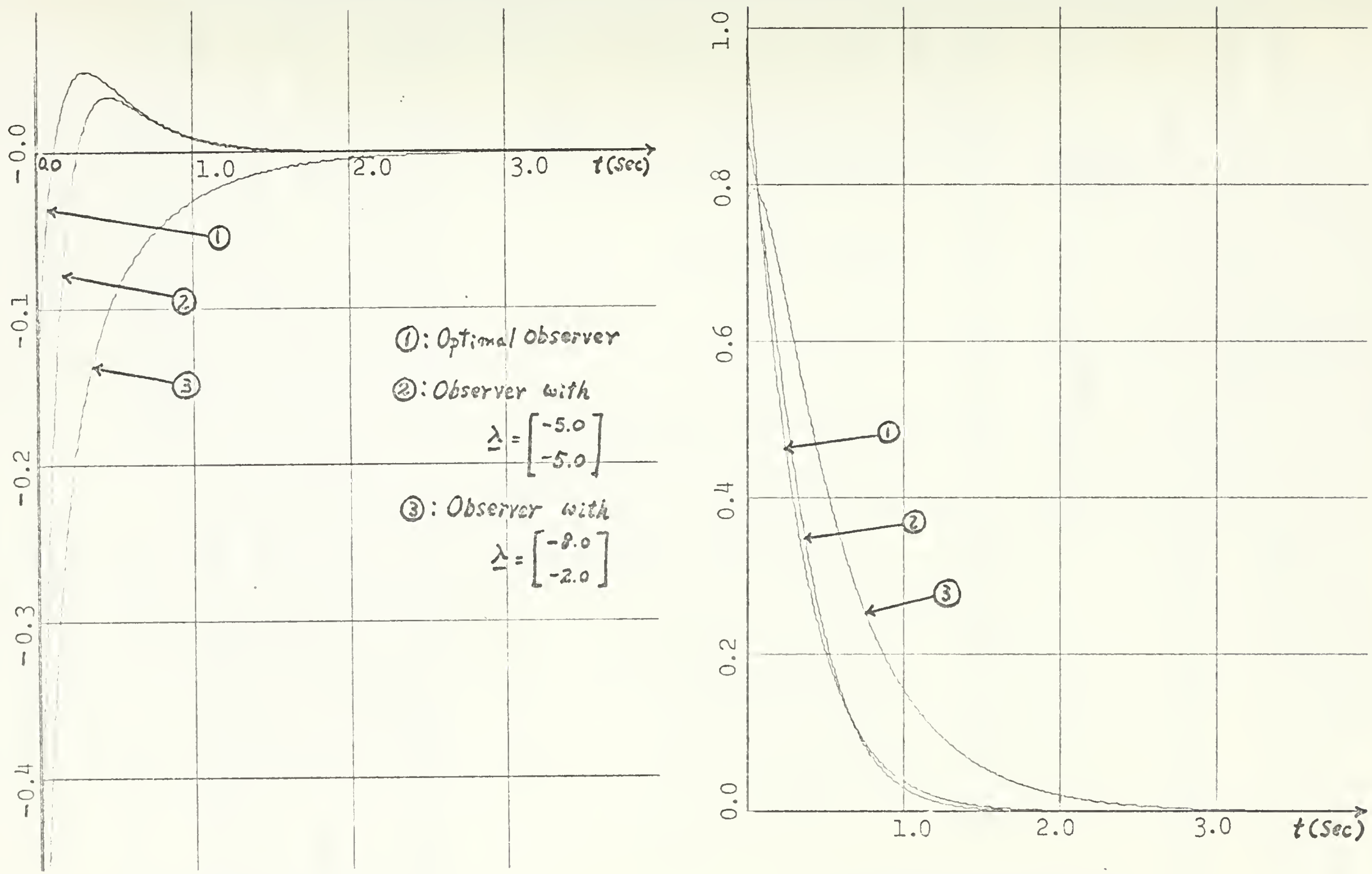
FIGURE 6. ACTUAL STATES AND ESTIMATES USING THE OPTIMAL GAINS FOR THE WORST INITIAL CONDITION



(a) ESTIMATE ERROR FOR x(1)

(b) ESTIMATE ERROR FOR x(2)

FIGURE 7. ESTIMATE ERRORS FOR DIFFERENT OBSERVERS FOR THE BEST INITIAL CONDITIONS



(a) ESTIMATE ERROR FOR x(1)

(b) ESTIMATE ERROR FOR x(2)

FIGURE 8. ESTIMATE ERRORS FOR DIFFERENT OBSERVERS FOR THE WORST INITIAL CONDITIONS

B. A REDUCED-ORDER OBSERVER

The purpose of this example is to illustrate the application of the algorithm to the design of a reduced-order observer for a third-order plant.

Consider the plant characterized by the transfer function

$$\frac{Y(S)}{U(S)} = \frac{1}{S^3 + 4S^2 + 5S + 2}$$

This plant can be represented by the state and output equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \qquad (82)$$

$$y(t) = [1 \quad 0 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \qquad (83)$$

This plant is a third-order system with a single output so a second-order observer with arbitrary eigenvalues can be constructed. The output matrix $\underline{C}$ already has the same form as the one developed in the reduced-order observer theory. Hence, it is not necessary to change the variables, that is, for this case the matrices

$$\underline{A}' = \underline{A} \quad , \quad \underline{B}' = \underline{B} \quad .$$

The matrices $\underline{A}$ and $\underline{B}$ can be partitioned as

$$\underline{A} = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ -2 & -5 & -4 \end{array}\right] \qquad \underline{B} = \left[\begin{array}{c} 0 \\ \hline 0 \\ 1 \end{array}\right]$$

Therefore, the reduced-order observer system matrix is

$$\underline{F} = \underline{A}_{22} - \underline{G}\,\underline{A}_{12} = \begin{bmatrix} -g_1 & 1 \\ -5-g_2 & -4 \end{bmatrix} \tag{84}$$

which has the corresponding characteristic equation

$$\lambda^2 + (4 + g_1)\lambda + 4g_1 + g_2 + 5 = 0 \tag{85}$$

Suppose one selects the starting observer eigenvalues at $\lambda = -3, -2$. This gives the characteristic equation

$$(\lambda + 3)(\lambda + 2) = \lambda^2 + 5\lambda + 6 = 0 \tag{86}$$

Equating the coefficients of the same powers of λ in (85) and (86) yields

$$\underline{G} = \begin{bmatrix} 1.0 \\ -3.0 \end{bmatrix}$$

This is the initial guessed value of the observer gain matrix $\underline{G}$. If the weighting matrix $\underline{Q}$ is chosen arbitrarily as $\underline{Q} = \underline{I}$, and equation (67) is solved for $\underline{V}$ the result is

$$V_{11} = \frac{36 + 4g_1 + 11g_2 + g_2^2}{40 + 42g_1 + 8g_2 + 2g_1g_2 + 8g_1^2}$$

$$V_{12} = \frac{4 - 5g_1 - g_1g_2}{40 + 42g_1 + 8g_2 + 2g_1g_2 + 8g_1^2} \tag{87}$$

$$V_{22} = \frac{6 + 4g_1 + g_2 + g_1^2}{40 + 42g_1 + 8g_2 + 2g_1g_2 + 8g_1^2}$$

Due to the symmetry of $\underline{V}$ only three equations are needed. The proposed algorithm gives the solution for the observer gain matrix as

$$\underline{G} = \begin{bmatrix} 26.0 \\ -1.73438 \end{bmatrix} \tag{88}$$

if the sum of the eigenvalues of $\underline{F}$ is limited to -30.0. This gain matrix $\underline{G}$ gives the observer eigenvalues

$$\underline{\lambda} = \begin{bmatrix} -25.851 \\ -4.149 \end{bmatrix} \tag{89}$$

To compare this optimally designed reduced-order observer with others having different observer eigenvalues, consider two observers with the eigenvalues

$$\underline{\lambda} = \begin{bmatrix} -15.0 \\ -15.0 \end{bmatrix} , \qquad \underline{\lambda} = \begin{bmatrix} -20.0 \\ -10.0 \end{bmatrix} \tag{90}$$

respectively. The corresponding observer gain matrices are

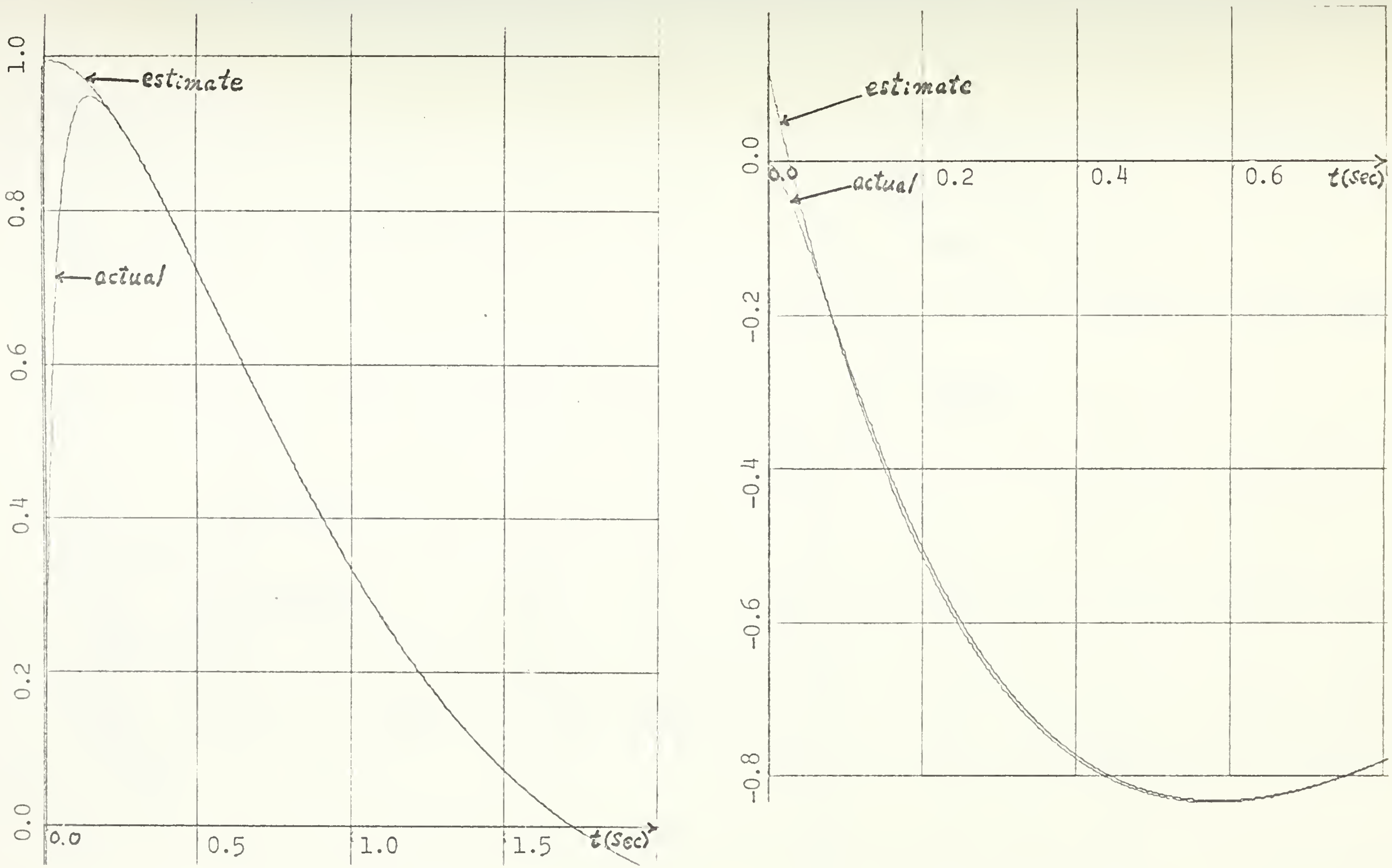
$$\underline{G} = \begin{bmatrix} 26.0 \\ 116.0 \end{bmatrix} , \qquad \underline{G} = \begin{bmatrix} 26.0 \\ 91.0 \end{bmatrix} \tag{91}$$

respectively. In Fig. 9 the actual and estimated states using the optimal gains are plotted for the best initial error conditions and in Fig. 10 for the worst initial error conditions. It is assumed that the initial error condition lies on the unit circle. The estimate errors for the three observers are plotted on one graph in Figs. 11 and 12 for the best and worst initial error conditions respectively. Table 3 shows the values of the performance measure for observers with different eigenvalues for the worst initial conditions.

C. OBSERVATIONS

The examples indicate that the observers designed by using the proposed technique avoid large errors at a cost of having relatively small errors that slowly approach zero. This is to be anticipated from the form of the performance measure selected.

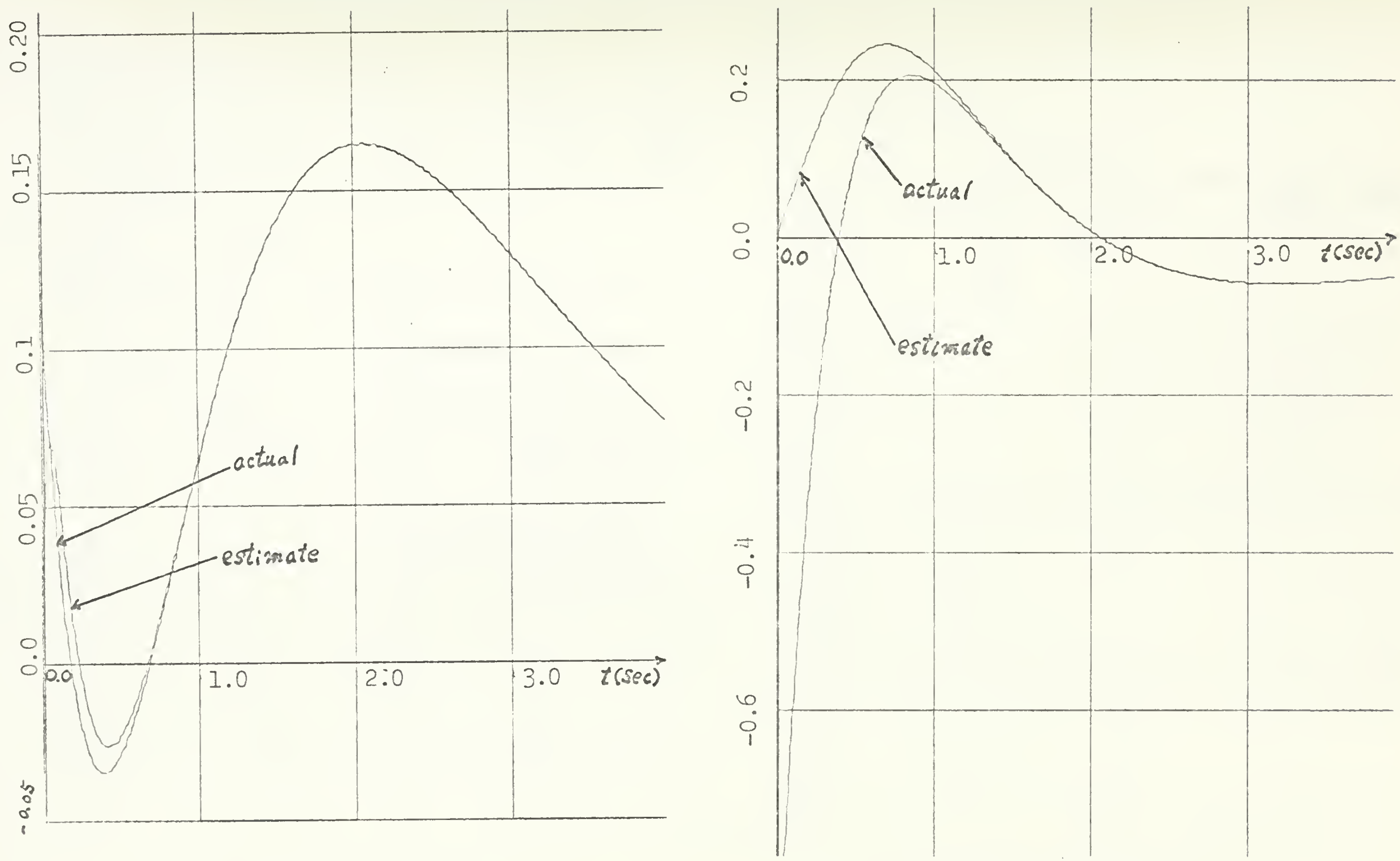
The design approach is, however, conservative in the sense that the resulting observers function effectively for



(a) x(2) AND ITS ESTIMATE

(b) x(3) AND ITS ESTIMATE

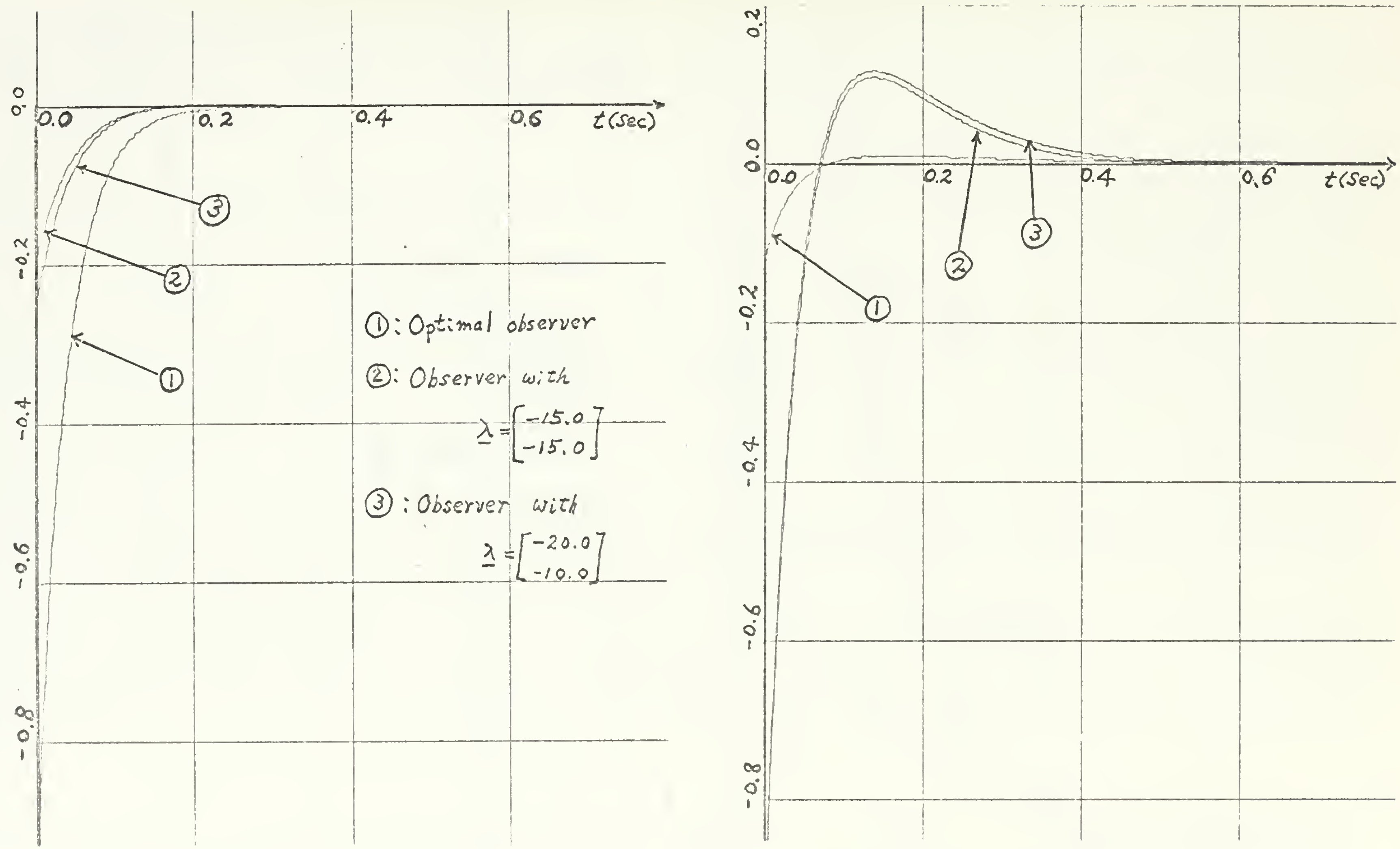
FIGURE 9 ACTUAL STATES AND ESTIMATES USING THE OPTIMAL GAINS FOR THE BEST INITIAL CONDITION



(a) x(2) AND ITS ESTIMATE

(b) x(3) AND ITS ESTIMATE

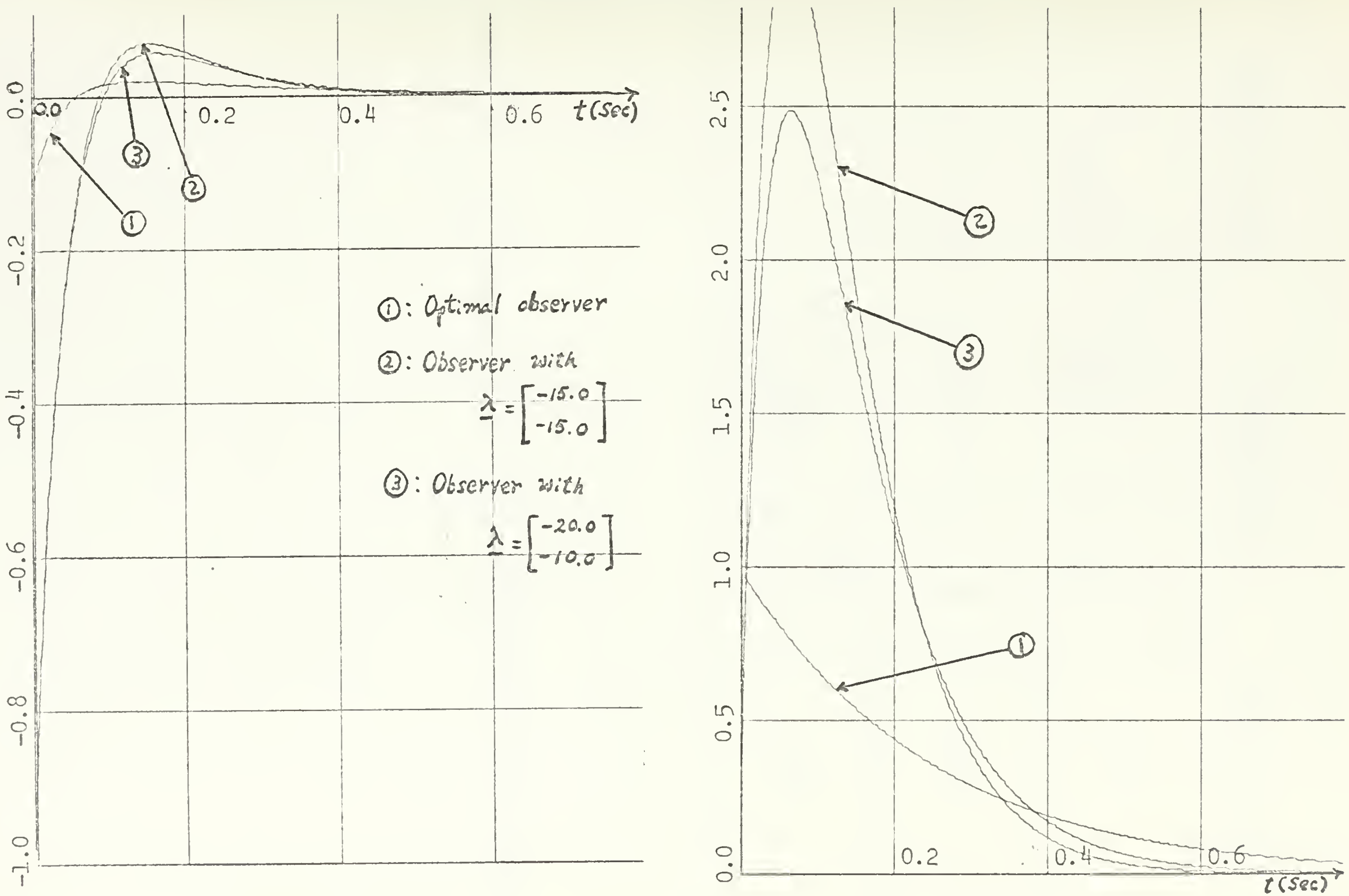
FIGURE 10. ACTUAL STATES AND ESTIMATES USING THE OPTIMAL GAINS FOR THE WORST INITIAL CONDITION



(a) ESTIMATE ERROR FOR X(2)

(b) ESTIMATE ERROR FOR X(3)

FIGURE 11. ESTIMATE ERRORS FOR DIFFERENT OBSERVER EIGENVALUES FOR THE BEST INITIAL CONDITIONS.



(a) ESTIMATE ERROR FOR x(2)

(b) ESTIMATE ERROR FOR x(3)

FIGURE 12. ESTIMATE ERRORS FOR DIFFERENT OBSERVER EIGENVALUES FOR THE WORST INITIAL CONDITIONS.

Observer Gains (Eigenvalues)	Worst Initial Conditions	Values of Performance Measure
$g_1 = 26.0$, $g_2 = -1.73438$ $\begin{pmatrix} \lambda_1 = -25.851 \\ \lambda_2 = -4.149 \end{pmatrix}$	$\underline{e}(0) = \begin{bmatrix} 0.122 \\ -0.993 \end{bmatrix}$	0.123
$g_1 = 26.0$, $g_2 = 116.0$ $\begin{pmatrix} \lambda_1 = -15.0 \\ \lambda_2 = -15.0 \end{pmatrix}$	$\underline{e}(0) = \begin{bmatrix} 0.98 \\ -0.2 \end{bmatrix}$	1.150
$g_1 = 26.0$, $g_2 = 91.0$ $\begin{pmatrix} \lambda_1 = -20.0 \\ \lambda_2 = -10.0 \end{pmatrix}$	$\underline{e}(0) = \begin{bmatrix} 0.966 \\ -0.26 \end{bmatrix}$	0.842

TABLE 3. PERFORMANCE VALUES FOR DIFFERENT OBSERVER EIGENVALUES

situations in which the initial conditions are worst. In exchange for this characteristic, performance for less difficult initial conditions is compromised.

V. CONCLUSIONS

In this thesis a new method for designing observers has been developed. The optimal observer gains are determined by solving a minimax problem. The proposed algorithm is easily suited to digital computation and especially applicable to higher-order systems.

The observers computed by using the algorithm have a tendency to avoid large errors at the expense of small errors which slowly decay. As anticipated, the observers designed by the proposed technique are efficient for the worst initial error conditions.

APPENDIX

DERIVATION OF EQUATIONS (29), (30), and (31)

For the system given by (22), one obtains the observer error equation

$$\dot{\underline{e}}(t) = \underline{F}\,\underline{e}(t) \tag{A.1}$$

where $\underline{e}(t)$ is the observer error and $\underline{F}$ is defined as

$$\underline{F} \triangleq \underline{A} - \underline{G}\,\underline{C}$$

Let $\underline{\Phi}(t)$ be the transition matrix of (A.1). The solution of (A.1) is given by

$$\underline{e}(t) = \underline{\Phi}(t)\,\underline{e}(0) \tag{A.2}$$

and $\underline{\Phi}(t)$ satisfies the differential equation

$$\frac{d}{dt}\,\underline{\Phi}(t) = \underline{F}\,\underline{\Phi}(t) \tag{A.3}$$

with the boundary condition

$$\underline{\Phi}(0) = \underline{I}$$

In addition, the final state is related to any preceding state by

$$\underline{e}(t_f) = \underline{\Phi}(t_f - \tau)\,\underline{e}(\tau) \tag{A.4}$$

and to the initial state by

$$\underline{e}(t_f) = \underline{\phi}(t_f)\ \underline{e}(0) \tag{A.5}$$

Differentiating (A.4) with respect to t gives

$$\dot{\underline{\phi}}(t_f - t) = -\ \underline{\phi}(t_f - t)\ \underline{F} \tag{A.6}$$

for all t_f and t with the boundary condition

$$\underline{\phi}(t_f - t_f) = \underline{\phi}(0) = \underline{I}$$

The performance measure for (A.1) is given by

$$J = \int_0^{t_f} [\ \underline{e}^T(\tau)\ \underline{Q}\ \underline{e}(\tau)\ ]\ d\tau \tag{A.7}$$

Substituting for $\underline{e}(\tau)$ using (A.2) gives

$$J = \int_0^{t_f} [\ \underline{e}^T(0)\ \underline{\phi}^T(\tau)\ \underline{Q}\ \underline{\phi}(\tau)\ \underline{e}(0)\ ]\ d\tau \tag{A.8}$$

Rearranging (A.8), one obtains

$$J = \underline{e}^T(0)\ [\ \int_0^{t_f} \underline{\phi}^T(\tau)\ \underline{Q}\ \underline{\phi}(\tau)\ d\tau\ ]\ \underline{e}(0) \tag{A.9}$$

Defining

$$\underline{V}(t) \triangleq \int_t^{t_f} \underline{\phi}^T(\tau - t)\ \underline{Q}\ \underline{\phi}(\tau - t)\ d\tau \tag{A.10}$$

then

$$\underline{V}(0) = \int_0^{t_f} \underline{\phi}^T(\tau)\ \underline{Q}\ \underline{\phi}(\tau)\ d\tau \tag{A.11}$$

Therefore, we have the final result (29)

$$J = \underline{e}^T(0)\ \underline{V}(0)\ \underline{e}(0) \tag{29}$$

Differentiating (A.10) with respect to t by using Leibnitz's rule

$$\dot{\underline{V}}(t) = \int_t^{t_f} [\ \dot{\underline{\phi}}^T(\tau - t)\ \underline{Q}\ \underline{\phi}(\tau - t) + \underline{\phi}^T(\tau - t)\ \underline{Q}\ \dot{\underline{\phi}}(\tau - t)\ ]\ d\tau$$

$$- \underline{\phi}^T(t - t)\ \underline{Q}\ \underline{\phi}(t - t)$$

$$= \int_t^{t_f} [\ \dot{\underline{\phi}}^T(\tau - t)\ \underline{Q}\ \underline{\phi}(\tau - t) + \underline{\phi}^T(\tau - t)\ \underline{Q}\ \dot{\underline{\phi}}(\tau - t)\ ]\ d\tau$$

$$- \underline{Q} \tag{A.12}$$

Substituting (A.6) for $\dot{\underline{\phi}}(\tau - t)$ gives

$$\dot{\underline{V}}(t) = \int_t^{t_f} \Big\{ - [\underline{F}^T\ \underline{\phi}^T(\tau - t)\ \underline{Q}\ \underline{\phi}(\tau - t)]$$

$$- [\underline{\phi}^T(\tau - t)\ \underline{Q}\ \underline{\phi}(\tau - t)\ \underline{F}] \Big\}\ d\tau\ -\ \underline{Q} \tag{A.13}$$

Rearranging (A.13) gives

$$\dot{\underline{V}}(t) = - \underline{F}^T\ [\ \int_t^{t_f} \underline{\phi}^T(\tau - t)\ \underline{Q}\ \underline{\phi}(\tau - t)\ d\tau]$$

$$- [\ \int_t^{t_f} \underline{\phi}^T(\tau - t)\ \underline{Q}\ \underline{\phi}(\tau - t)\ d\tau]\ \underline{F} - \underline{Q} \tag{A.14}$$

Substituting for $\underline{V}(t)$ using (A.10) gives equation (30)

$$\dot{\underline{V}}(t) = -\underline{F}^T \underline{V}(t) - \underline{V}(t)\,\underline{F} - \underline{Q} \tag{30}$$

Letting $t = t_f$ in (A.10) gives equation (31)

$$\underline{V}(t_f) = \underline{0} \tag{31}$$

Taking the transpose of (A.10), the right side does not change, hence $\underline{V}(t)$ is symmetric.

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A new technique for observer design which requires the solution of a minimax problem is presented. An algorithm for solving the minimax problem is proposed and applied to the design of full-order and reduced-order observers.

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